

5. $f(x) = x^4 + 3x^2 \Rightarrow f'(x) = 4x^3 + 6x$, so $f(-1) = 4$ and $f'(-1) = -10$.

Thus, $L(x) = f(-1) + f'(-1)(x - (-1)) = 4 + (-10)(x + 1) = -10x - 6$.

7. $f(x) = \cos x \Rightarrow f'(x) = -\sin x$, so $f(\frac{\pi}{2}) = 0$ and $f'(\frac{\pi}{2}) = -1$.

Thus, $L(x) = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) = 0 - 1(x - \frac{\pi}{2}) = -x + \frac{\pi}{2}$.

15. To estimate $(2.001)^5$, we'll find the linearization of $f(x) = x^5$ at $a = 2$. Since $f'(x) = 5x^4$, $f(2) = 32$, and $f'(2) = 80$,

we have $L(x) = 32 + 80(x - 2) = 80x - 128$. Thus, $x^5 \approx 80x - 128$ when x is near 2, so

$$(2.001)^5 \approx 80(2.001) - 128 = 160.08 - 128 = 32.08.$$

17. To estimate $(8.06)^{2/3}$, we'll find the linearization of $f(x) = x^{2/3}$ at $a = 8$. Since $f'(x) = \frac{2}{3}x^{-1/3} = 2/(3\sqrt[3]{x})$,

$f(8) = 4$, and $f'(8) = \frac{1}{3}$, we have $L(x) = 4 + \frac{1}{3}(x - 8) = \frac{1}{3}x + \frac{4}{3}$. Thus, $x^{2/3} \approx \frac{1}{3}x + \frac{4}{3}$ when x is near 8, so

$$(8.06)^{2/3} \approx \frac{1}{3}(8.06) + \frac{4}{3} = \frac{12.06}{3} = 4.02.$$

20. If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope 6. If the change in x is 0.01, the change in y on the tangent line is 0.06, and approximating $(1.01)^6$ with 1.06 is reasonable.

24. (a) For $y = f(t) = e^{\tan \pi t}$, $f'(t) = e^{\tan \pi t} \cdot \sec^2(\pi t) \cdot \pi$, so $dy = \pi \sec^2(\pi t) e^{\tan \pi t} dt$.

(b) For $y = f(z) = \sqrt{1 + \ln z}$, $f'(z) = \frac{1}{2}(1 + \ln z)^{-1/2} \cdot \frac{1}{z}$, so $dy = \frac{1}{2z\sqrt{1 + \ln z}} dz$.

28. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30 \text{ cm}^2$.

(b) Relative error $= \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2 dr}{r} = \frac{2(0.2)}{24} = \frac{0.2}{12} = \frac{1}{60} = 0.01\bar{6}$.

Percentage error $= \text{relative error} \times 100\% = 0.01\bar{6} \times 100\% = 1.\bar{6}\%$.

36. (a) $g'(x) = \sqrt{x^2 + 5} \Rightarrow g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.

$$g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85.$$

(b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g . Hence, the estimates in part (a) are too small.

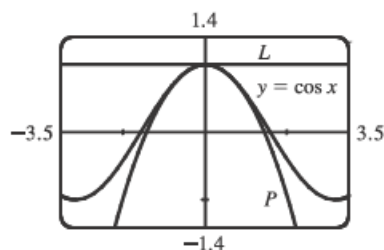
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

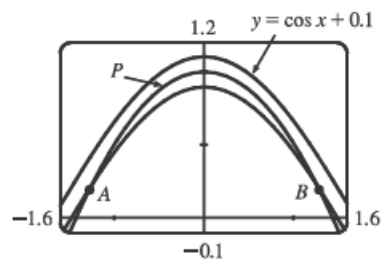
$$\begin{aligned} P(0) = f(0): & \quad A = \cos 0 = 1 \\ P'(0) = f'(0): & \quad B = -\sin 0 = 0 \\ P''(0) = f''(0): & \quad 2C = -\cos 0 = -1 \Rightarrow C = -\frac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $|\cos x - (1 - \frac{1}{2}x^2)| < 0.1 \Leftrightarrow -0.1 < \cos x - (1 - \frac{1}{2}x^2) < 0.1 \Leftrightarrow 0.1 > (1 - \frac{1}{2}x^2) - \cos x > -0.1 \Leftrightarrow \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \Leftrightarrow \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B . Zooming in or using an intersect feature, we find that the x -coordinates of B and A are about ± 1.26 . Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x - a) + C(x - a)^2$, then $P'(x) = B + 2C(x - a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$\begin{aligned} P(a) = f(a): & \quad A = f(a) \\ P'(a) = f'(a): & \quad B = f'(a) \\ P''(a) = f''(a): & \quad 2C = f''(a) \Rightarrow C = \frac{1}{2}f''(a) \end{aligned}$$

Thus, $P(x) = A + B(x - a) + C(x - a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$.

4. From Example 2 in Section 3.9, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and

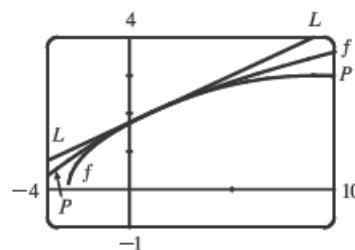
$$f'(x) = \frac{1}{2}(x+3)^{-1/2}. \text{ So } f''(x) = -\frac{1}{4}(x+3)^{-3/2} \Rightarrow f''(1) = -\frac{1}{32}.$$

From Problem 3, the quadratic approximation $P(x)$ is

$$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2.$$

The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear

approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.



	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373 ...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118 ...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015 ...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_n(x-a)^n$. If we put $x = a$ in this equation,

then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain

$$T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots + nc_n(x-a)^{n-1}. \text{ Substituting } x = a \text{ gives } T_n'(a) = c_1.$$

Differentiating again, we have $T_n''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \dots + (n-1)nc_n(x-a)^{n-2}$ and so

$$T_n''(a) = 2c_2. \text{ Continuing in this manner, we get } T_n'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + \dots + (n-2)(n-1)nc_n(x-a)^{n-3}$$

and $T_n'''(a) = 2 \cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$T_n^{(4)}(a) = 2 \cdot 3 \cdot 4c_4 \text{ and in general, for any integer } k \text{ between 1 and } n, T_n^{(k)}(a) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot kc_k = k! c_k \Rightarrow$$

$$c_k = \frac{T_n^{(k)}(a)}{k!}. \text{ Because we want } T_n \text{ and } f \text{ to have the same derivatives at } a, \text{ we require that } c_k = \frac{f^{(k)}(a)}{k!} \text{ for}$$

$$k = 1, 2, \dots, n.$$

6. $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{array}{ll} f(x) = \cos x & f(0) = \cos 0 = 1 \\ f'(x) = -\sin x & f'(0) = -\sin 0 = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \\ f^{(4)}(x) = \cos x & f^{(4)}(0) = 1 \end{array}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$.

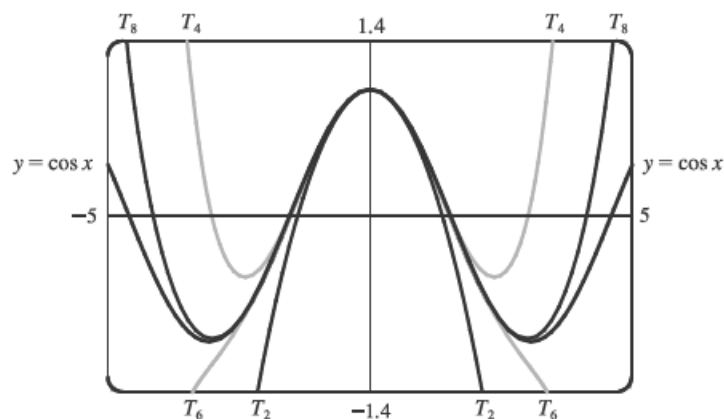
From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2 , T_4 , and T_6 consist of the

initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and

$T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. We graph T_2 , T_4 , T_6 , T_8 , and f :



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.