Math 210 Midterm 2 Answers

Problem 1 (5 pts) *Compute the following limit.*

$$\lim_{x\to 2}(10x+7)$$

We can just plug in $x=2$ here:

$$10\cdot 2+7=27$$

Problem 2 (5 pts) *Compute the following limit.*

$$\lim_{x\to 1}\frac{x^2-1}{x-1}$$

Since plugging in gets $0/0$, we must first factor and cancel:

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\\ &= \lim_{x\to 1}\frac{x+1}{1}\\ &= 1+1=2\end{aligned}$$

Problem 3 (5 pts) *Compute the following limit.*

$$\lim_{x\to\infty}\frac{3x^2+5}{2x^2+9}$$

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2+5}{2x^2+9} &= \lim_{x\to\infty}\frac{3x^2+5}{2x^2+9}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{3+\frac{5}{x^2}}{2+\frac{9}{x^2}}\\ &= \frac{3+0}{2+0}\\ &= \frac{3}{2}\end{aligned}$$

Problem 4 (5 pts) *Compute the following limit.*

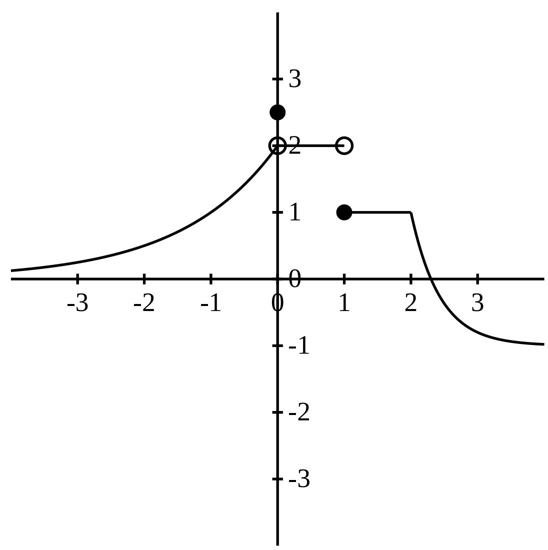
$$\lim_{x\to\infty}\frac{7}{2+(0.5)^x}$$

Since $\lim_{x\to\infty}(0.5)^x=0$,

$$\lim_{x\to\infty}\frac{7}{2+(0.5)^x} = \frac{7}{2+0}=\frac{7}{2}$$

Problem 5 (10 pts) *Draw a graph of a function $f(x)$ with the following features:*

$$\begin{aligned}
& && f \text{ is defined everywhere} \\
& && f \text{ is continuous everywhere except at } x = 0 \text{ and } x = 1 \\
& && f \text{ is differentiable everywhere except at } x = 0,\ x = 1,\ \text{and } x = 2. \\
\lim_{x\to 0} f(x) &= && 2 \\
\lim_{x\to 1^-} f(x) &= && 2 \\
\lim_{x\to 1} f(x) & && \text{does not exist} \\
f(1) &= && 1 \\
\lim_{x\to -\infty} f(x) &= && 0 \\
\lim_{x\to +\infty} f(x) &= && -1
\end{aligned}$$



Note the following features of this graph that are logical consequences of the conditions given:

- *The left limit and right limit at $x = 0$ match, so that the limit exists, and is equal to 2; but since f is not continuous there, and since f is defined everywhere, the actual value of $f(0)$ is something else.*
- *The left and right limits do not agree at $x = 1$. This is so that this limit does not exist.*
- *Note the corner at $x = 2$. This is so that the function is not differentiable but continuous here. A vertical tangent is also acceptable.*

Problem 6 (5 pts) *Below are some facts about $f(x)$ and $g(x)$. Determine*

$$\lim_{x \to 1} f(x)g(x)$$

$$\begin{aligned} f(0) &= 2 \\ f(1) &= 3 \\ f(2) &= 6 \\ g(0) &= 1 \\ g(1) &= 5 \\ g(2) &= 9 \\ \lim_{x \to 1} f(x) &= 10 \\ \lim_{x \to 1} g(x) &= 7 \end{aligned}$$

Since

$$\lim_{x \to 1} f(x)g(x) = \lim_{x \to 1} f(x) \cdot \lim_{x \to 1} g(x)$$

assuming the limits exist, then we conclude the answer is

$$10 \cdot 7 = 70$$

Problem 7 (5 pts) *Find the slope of the tangent line of $y = x^2 + 2x$ at $x = 1$. You must show all your work and use only methods described in class so far. Just writing down the answer is worth no points.*

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{((1+h)^2 + 2(1+h)) - (1^2 + 2 \cdot 1)}{h} \\ &= \lim_{h \to 0} \frac{(1 + 2h + h^2 + 2 + 2h) - 3}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2 + 2h}{h} \\ &= \lim_{h \to 0} \frac{h(2 + h + 2)}{h} \\ &= \lim_{h \to 0} \frac{2 + h + 2}{1} \\ &= \lim_{h \to 0} (4 + h) = 4 + 0 = 4 \end{aligned}$$

Problem 8 (5 pts) *As a rocket is used, its fuel is being expended. The amount of fuel in the rocket in liters is $A(t)$, a function of time, t, measured in seconds since the rocket starts up. Write down a mathematical formula involving the function $A(t)$ that says the following: at exactly $t = 10$ seconds after the rocket starts up, fuel is being expended at a rate of 300 liters per second.*

$$A'(10) = -300 \text{ liters per second}$$

The rate of change is the derivative at the time indicated ($t = 10$), and the rate of change is -300 liters per second.

Problem 9 (5 pts) *A knob on an amplifier is used to control the volume of sound coming from an MP3 player. Let x be the position of the knob (marked from 1 to 11) and let V be the volume of sound in decibels. Explain, in your own words, the meaning of the statement*

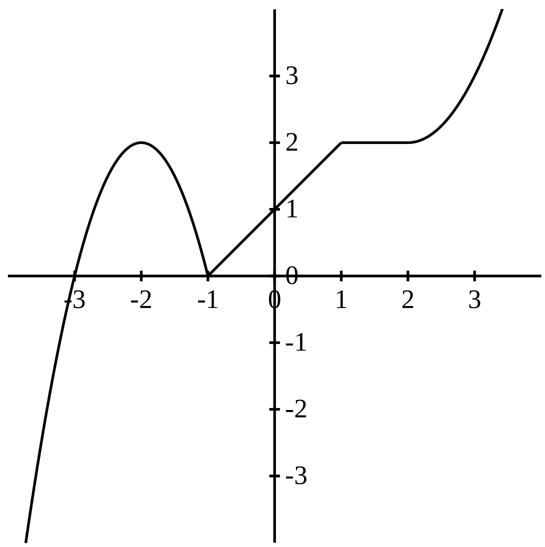
$$V'(5) = 20$$

If the knob is at 5, a small increase in knob position will result in a volume increase of 20 decibels for every unit increase in knob position.

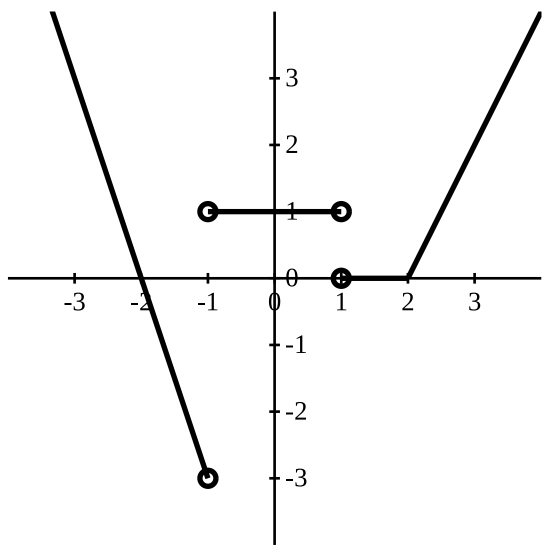
Note that if you used the phrase "rate of change" without qualification, the answer is not quite correct, since nothing says that you will actually turn the knob. Rather, the derivative describes the effect on volume of a hypothetical change in knob position.

Problem 10 (10 pts) *Below is a graph of f. Sketch a graph of f'.*

f:

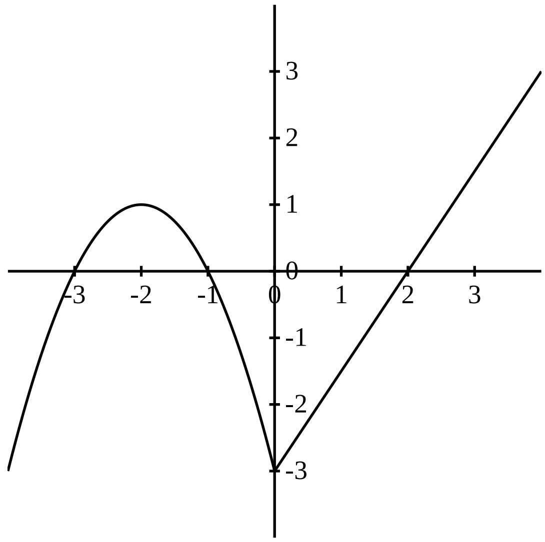


f':



Problem 11 (5 pts) *Suppose we are interested in a certain function $f(x)$. Its derivative is given in the graph below. State the interval that describes where $f(x)$ is increasing.*

f':



f is increasing where f' is positive, so we read on the graph where f' is positive: from -3 to 1 and when $x > 2$.

Problem 12 (10 pts) *True or False?*

TRUE *If $f(x)$ is differentiable at a then $f(x)$ is continuous at a.* **This is explained in the discussion on differentiability.**

FALSE *If $f(x)$ is continuous at a then $f(x)$ is differentiable at a.* **The absolute value function $f(x) = |x|$ is continuous everywhere but not differentiable at $x = 0$.**

TRUE *If $f(x)$ is continuous at a then $f(a)$ must be defined.* **One of the requirements of continuity, section 2.4**

TRUE *For $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ to work, $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ must exist.* **This is the correct criterion for this limit law, section 2.3**

FALSE *The fraction $0/0$ simplifies to 1.* **No, $0/0$ is undefined.**

TRUE *If $0 \le f(x) \le x$, then $\lim_{x\to 0} f(x) = 0$.* **This follows from the squeeze theorem.**

TRUE *The derivative of $f(x)$ at $x = a$ is $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.* **This is the definition of the derivative, section 2.6**

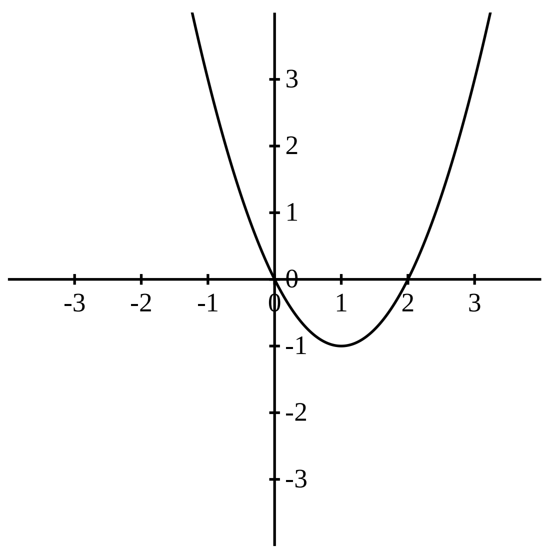
TRUE *If $f'(x)$ is always positive, then $f(x)$ is always increasing.* **This is how we interpret the derivative being positive.**

FALSE *If $f'(x)$ is increasing at $x = 2$ then $f(x)$ is increasing at $x = 2$.* f' **increasing just means** f **is concave up there.**

FALSE *If $B(x)$ has units of pounds, and x has units of inches, then $B'(x)$ has units of pounds.* **No, it is pounds per inch.**

Problem 13 (5 pts) *Sketch a graph of a function $f(x)$ with the following properties:*

$$\begin{aligned} f'(x) &> 0 \text{ if } x > 1 \\ f'(x) &< 0 \text{ if } x < 1 \\ f''(x) &> 0 \text{ for all } x \\ f(0) &= 0 \end{aligned}$$



Problem 14 (5 pts) *What does it mean to say that $f(x)$ is continuous at a?*

It means that

- $f(a)$ is defined,
- $\lim_{x \to a} f(x)$ exists, and
- $\lim_{x \to a} f(x) = f(a)$.

Problem 15 (5 pts) *Find the values of a and b that would make f(x) continuous.*

$$f(x) = \begin{cases} 5x, & x < 3 \\ a, & x = 3 \\ 2 - bx, & x > 3 \end{cases}$$

For $f(x)$ to be continuous, the left limit, right limit, and value at $x = 3$ must coincide. So we compute

$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} 5x = 5 \cdot 3 = 15 \\ \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (2 - bx) = 2 - b3 \\ f(3) &= a \end{aligned}$$

Therefore $a = 15$, and $2 - 3b = 15$. Solving the second equation for b, we get

$$b = -\frac{13}{3}.$$

Problem 16 (5 pts) *If $f(x)$ is continuous everywhere, and $f(1) = 6$ and $f(2) = 10$, why can we conclude $f(x) = 7$ for some value of x? What can we say about the value of x?*

Since $f(x)$ is continuous on the interval $[1, 2]$, the intermediate value theorem guarantees the existence of at least one x in the interval $[1, 2]$ that has $f(x) = 7$.

Problem 17 (5 pts) *Find a formula for a function $f(x)$ that satisfies the following conditions (a graph will not be accepted for this problem).*

$$\begin{aligned} \lim_{x\to 1} f(x) &= -\infty \\ \lim_{x\to \infty} f(x) &= 2 \\ \lim_{x\to -\infty} f(x) &= 2 \end{aligned}$$

To get the vertical asymptote at $x = 1$ we first take x^{-2}, then flip it vertically and shift it one unit to the right: $-\frac{1}{(x-1)^2}$. Then to make the horizontal asymptotes correct we need to shift the graph up by 2, and so we get

$$f(x) = -\frac{1}{(x-2)^2} + 2$$