Math 210 Midterm 3 Answers

Problem 1 (2 pts) *Find the derivative.*

$$x^5 + x^3 + x^2 + 5$$

$$5x^4 + 3x^2 + 2x$$

Problem 2 (3 pts) *Find the derivative.*

$$7^x$$

$$7^x \cdot \ln 7$$

Problem 3 (3 pts) *Find the derivative.*

$$\cos x + \tan x$$

$$-\sin x + \sec^2 x$$

Problem 4 (2 pts) *Find the derivative.*

$$\sqrt{x}$$

$$\frac{1}{2}x^{-1/2}$$

Problem 5 (2 pts) *Find the derivative.*

$$x^e$$

$$ex^{e-1}$$

Problem 6 (3 pts) *Find the derivative.*

$$\ln(e) + \sin(3)$$

$$0$$

Problem 7 (2 pts) *Find the derivative.*

$$\sqrt{x^3 + 5x + 2}$$

$$\frac{1}{2}(x^3 + 5x + 2)^{-1/2}(3x^2 + 5)$$

Problem 8 (3 pts) *Find the derivative.*

$$\ln \sin x - 2\sin^2 x$$

$$\frac{1}{\sin x}\cos x - 4\sin x\cos x$$

Problem 9 (5 pts) *Find the derivative.*

$$x^3 \tan x$$

$$x^3 \sec^2 x + 3x^2 \tan x$$

Problem 10 (5 pts) *Find the derivative.*

$$\sec(\sec x)$$

$$\sec(\sec x)\tan(\sec x)\sec x\tan x$$

Problem 11 (5 pts)

$$e^{2t}\sec^{-1} t + \ln|t|$$

$$e^{2t} 2\sec^{-1} t + e^{2t}\frac{1}{|t|\sqrt{t^2-1}} + \frac{1}{t}$$

Problem 12 (5 pts) *Find the derivative.*

$$\sin^{-1}(x + e^x)$$

$$\frac{1}{\sqrt{1-(x+e^x)^2}}(1+e^x)$$

Problem 13 (5 pts) *Find the derivative.*

$$x^x$$

We use logarithmic differentiation:

$$\begin{aligned}
y &= x^x \\
\ln y &= \ln(x^x) \\
\ln y &= x\ln x \\
\frac{d}{dx}\ln y &= \frac{d}{dx}(x\ln x) \\
\frac{1}{y}\frac{dy}{dx} &= x\frac{1}{x} + 1\cdot\ln x \\
\frac{1}{y}\frac{dy}{dx} &= 1 + \ln x \\
\frac{dy}{dx} &= y(1+\ln x) \\
\frac{dy}{dx} &= x^x(1+\ln x)
\end{aligned}$$

Problem 14 (5 pts) *Find the derivative of the function described implicitly (y is a function of x).*

$$x^3 + y^5 + 2x^3 \cos y = 3$$

$$\begin{aligned}
\frac{d}{dx}(x^3 + y^5 + 2x^3 \cos y) &= \frac{d}{dx}(3) \\
3x^2 + 5y^4 \frac{dy}{dx} + 6x^2 \cos y + 2x^3(-1) \sin y \frac{dy}{dx} &= 0 \\
5y^4 \frac{dy}{dx} + 2x^3(-1) \sin y \frac{dy}{dx} &= -3x^2 - 6x^2 \cos y \\
\frac{dy}{dx}(5y^4 - 2x^3 \sin y) &= -3x^2 - 6x^2 \cos y \\
\frac{dy}{dx} &= \frac{-3x^2 - 6x^2 \cos y}{5y^4 - 2x^3 \sin y}
\end{aligned}$$

Problem 15 (5 pts) *Use logarithmic differentiation to find the derivative.*

$$\frac{x^5 e^x}{(3x+1)\sqrt{4x+5}}$$

$$\begin{aligned}
y &= \frac{x^5 e^x}{(3x+1)\sqrt{4x+5}} \\
\ln y &= \ln\left(\frac{x^5 e^x}{(3x+1)\sqrt{4x+5}}\right) \\
\ln y &= \ln x^5 + \ln e^x - \ln(3x+1) - \ln\sqrt{4x+5} \\
\ln y &= 5\ln x + x - \ln(3x+1) - \frac{1}{2}\ln(4x+5) \\
\frac{d}{dx}\ln y &= \frac{d}{dx}\left(5\ln x + x - \ln(3x+1) - \frac{1}{2}\ln(4x+5)\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{5}{x} + 1 - \frac{3}{3x+1} - \frac{2}{4x+5} \\
\frac{dy}{dx} &= y\left(\frac{5}{x} + 1 - \frac{3}{3x+1} - \frac{2}{4x+5}\right) \\
\frac{dy}{dx} &= \frac{x^5 e^x}{(3x+1)\sqrt{4x+5}}\left(\frac{5}{x} + 1 - \frac{3}{3x+1} - \frac{2}{4x+5}\right)
\end{aligned}$$

Problem 16 (10 pts) *For the following, assume $f'(x) = g(x)$ and $g'(x) = h(x)$.*
True or False:

FALSE *The derivative of $f(x)g(x)$ is $g(x)h(x)$.* **No, by the product rule it should be $f'(x)g(x) + f(x)g'(x)$.**

TRUE *The derivative of $f(g(x))$ is $g(g(x))h(x)$.* **This is the chain rule.**

FALSE *The derivative of $g(f(x))$ is $h(g(x))$.* **No, you should use the chain rule to get $h(f(x))g(x)$.**

TRUE *We should expect that $f(x)$ will be fairly close to $f(5)+g(5)(x-5)$ when x is close enough to 5.* **This is the principle of the local linear approximation.**

FALSE *When x is very far from 5, we should expect $f(x)$ to be close to $f(5)+g(5)(x-5)$.* **The local linear approximation should work near the point where the approximation is based, but is not expected to work well far away from that point.**

Problem 17 (5 pts) *Prove the differentiation rule for $\tan^{-1} x$.*

$$\begin{aligned}
y &= \tan^{-1} x \\
\tan y &= x \\
\frac{d}{dx}\tan y &= \frac{d}{dx}(x) \\
\sec^2 y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\
\frac{dy}{dx} &= \frac{1}{1+\tan^2 y} \\
\frac{dy}{dx} &= \frac{1}{1+x^2}
\end{aligned}$$

Problem 18 (5 pts) *The energy of a particle is given by*

$$E = \frac{1}{2}mv^2$$

where m is the mass of the particle, which is known to be 4 kg, and v is the speed of the particle which is measured at 3.0 m/s, with an error of 0.2 m/s. Find the energy of the particle, with its corresponding error.

First,

$$E = \frac{1}{2}(4\text{ kg})(3.0\text{ m/s})^2 = 18\text{ kg } m^2/s^2.$$

The error is given by the differential:

$$dE = \frac{1}{2}m(2v)dv = mv\,dv$$

In our case, $m = 4$ kg, $v = 3.0$ m/s, and $dv = 0.2$ m/s, so

$$dE = 4\cdot 3\cdot 0.2\text{kg } m^2/s^2 = 2.4\text{ kg } m^2/s^2.$$

The energy is thus

$$(18 \pm 2.4)kg\ m^2/s^2$$

Problem 19 (5 pts) *Use a local linear approximation to* $\sqrt{x}$ *near* $x = 25$ *to estimate* $\sqrt{22}$.

For $f(x) = \sqrt{x} = x^{1/2}$, we have $f'(x) = \frac{1}{2}x^{-1/2}$. At $x = 25$ we get $f(25) = \sqrt{25} = 5$ and $f'(25) = \frac{1}{2}(25)^{-1/2} = \frac{1}{10} = 0.1$. To approximate $\sqrt{22}$ we consider that 22 is 3 less than 25, so $\Delta x = -3$. We get $\Delta f \cong f'(25)\Delta x = 0.1 \cdot -3 = -0.3$.

Thus, $\sqrt{22} \cong f(25) - 0.3 = 5 - 0.3 = 4.7$.

Problem 20 (3 pts) *Find the differential of*

$$x^3e^{3y}$$

$$3x^2\,dx\,e^{3y} + x^3e^{3y}3\,dy$$

Problem 21 (4 pts) *Find* $P(x)$*, the second-order Taylor polynomial for* e^x *at* $x = 0$.

If we write

$$P(x) = A + Bx + Cx^2$$

to find A, B, and C, we take derivatives:

$$\begin{aligned} P'(x) &= B + 2Cx \\ P''(x) &= 2C \end{aligned}$$

Now plug in $x = 0$:

$$\begin{aligned} P(0) &= A \\ P'(0) &= B \\ P''(0) &= 2C \end{aligned}$$

We compare this to what we get when we do the same for $f(x) = e^x$:

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \end{aligned}$$

and we again plug in $x = 0$:

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \end{aligned}$$

Comparing the values of f with those of P and their derivatives we match them up:

$$\begin{aligned} A &= 1 \\ B &= 1 \\ 2C &= 1 \end{aligned}$$

and conclude $A = 1$, $B = 1$, and $C = 1/2$. Therefore we get

$$P(x) = 1 + x + \frac{1}{2}x^2$$

Problem 22 (3 pts) *Explain what $P(x)$ is supposed to accomplish; in other words, why we might bother to compute it. (A one- or two-sentence answer is all that is expected here, not a long essay.)*

The function $P(x)$ is a quadratic function that gives the values of e^x when x is close to 0, better than the linear approximation would, because it matches the values of the function and two of its derivatives at $x = 0$.

Problem 23 (5 pts) *Some gas is enclosed in the lung of an animal. The pressure of the gas, P, in atmospheres, is given by the formula*

$$P = \frac{T}{V}$$

where T is the temperature in Kelvin and V is the volume in milliliters.

Currently the temperature is at 298 Kelvin, increasing at a rate of 2 Kelvin per minute. The volume is at 300 milliliters, increasing at a rate of 500 milliliters per minute. What is the instantaneous rate of change of the pressure currently?

We use the quotient rule:

$$\frac{dP}{dt} = \frac{VT' - TV'}{V^2}$$

and plug in $T = 298$, $T' = 2$, $V = 300$, $V' = 500$:

$$P' = \frac{(300)(2) - (298)(500)}{300^2} = -1.65 \text{ atmospheres per minute}$$

Problem 24 (5 pts) *Suppose $f(x)$ is a function, and its derivative satisfies the equation*

$$f'(x) = \sqrt{f(x)^3 + 5f(x) + 2}$$

Find the derivative of $f^{-1}(x)$.

$$\begin{aligned} y &= f^{-1}(x) \\ f(y) &= x \\ f'(y)\,y' &= 1 \\ y' &= \frac{1}{f'(y)} \\ y' &= \frac{1}{\sqrt{f(y)^3 + 5f(y) + 2}} \\ y' &= \frac{1}{\sqrt{x^3 + 5x + 2}} \end{aligned}$$