Math 210 Midterm 4 Answers

Problem 1 (5 pts) *Find the general antiderivative:*

$$f'(x) = 4x^5 + \sin x$$

$$f(x) = x^5 - \cos x + C$$

Problem 2 (5 pts) *Find the general antiderivative:*

$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2}$$

$$f(x) = \ln|x| + \tan^{-1}(x) + C$$

Problem 3 (5 pts) *Find the limit.*

$$\lim_{x\to 0} \frac{5x}{\tan(3x)}$$

Plugging in we get the indeterminate form $\frac{0}{0}$, so we use L'hospital:

$$\begin{aligned} \lim_{x\to 0} \frac{5x}{\tan(3x)} &= \lim_{x\to 0} \frac{5}{\sec^2(3x)\cdot 3} \\ &= \frac{5}{\sec^2(0)\cdot 3} \\ &= \frac{5}{1\cdot 3} = 5/3 \end{aligned}$$

Problem 4 (5 pts) *Find the limit.*

$$\lim_{x\to 0} \frac{x^2}{\cos x}$$

Plugging in, we get

$$\frac{0^2}{\cos 0} = \frac{0}{1} = 0.$$

Problem 5 (5 pts) *Find the limit.*

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{\sin x}{x^2} \right)$$

We combine the terms into one fraction:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{\sin x}{x^2}\right) &= \lim_{x\to 0}\left(\frac{x}{x^2}-\frac{\sin x}{x^2}\right)\\ &= \lim_{x\to 0}\frac{x-\sin x}{x^2}\end{aligned}$$

Plugging in we get the indeterminate form $\frac{0}{0}$, so we use L'hospital:

$$\lim_{x\to 0}\frac{1-\cos x}{2x}$$

Again, plugging in we get $\frac{0}{0}$, so we do L'hospital again:

$$\begin{aligned}&= \lim_{x\to 0}\frac{\sin x}{2}\\ &= \frac{\sin 0}{2}=0\end{aligned}$$

Problem 6 (5 pts) *Newton's method is being used to find the solution of*

$$x^3+4x+6=0$$

The process has resulted in $x_3=1$. Find the result of the next iteration.

First we identify the function:

$$f(x)=x^3+4x+6$$

and its derivative:

$$f'(x)=3x^2+4.$$

Using

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

we get

$$\begin{aligned}x_4 &= x_3-\frac{f(x_3)}{f'(x_3)}\\ &= 1-\frac{1^3+4\cdot 1+6}{3\cdot 1^2+4}\\ &= 1-\frac{11}{7}\\ &= -\frac{4}{7}\end{aligned}$$

Problem 7 (10 pts) *A college student's pile of laundry gathers in a mound that is in the shape of a cone where the height and radius of the cone are the same size. As more clothes are thrown on the pile the mound grows into a bigger cone, and the height and radius continue to be equal. The height is now 3 feet. The volume is increasing at a rate of 2 cubic feet per week. How quickly is the height of the pile changing?*

Let V be the volume of clothes in cubic feet, and h the height of the pile, and r the radius of the base. The formula for the volume of the cone is

$$V = \frac{1}{3}\pi r^2 h$$

and since all throughout, $r = h$, we can substitute:

$$V = \frac{1}{3}\pi h^2 h = \frac{1}{3}\pi h^3$$

We take the derivative:

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$$

and note that $\frac{dV}{dt} = 2$, and $h = 3$. So

$$2 = \pi(3^2)\frac{dh}{dt}$$

Solving, we get

$$\frac{dh}{dt} = \frac{2}{9\pi}$$

so the height is growing at a rate of $\frac{2}{9\pi}$ feet per week.

Problem 8 (5 pts) *True or False:*

TRUE *If a function has a relative maximum at c and $f'(c)$ is defined, then $f'(c) = 0$.* **This is Fermat's principle.**

FALSE *Every critical point is either a relative minimum or a relative maximum.* **It could be neither, such as the case of $y = x^3$ at $x = 0$.**

FALSE *Every function has a global maximum and a global minimum.* **No, $y = x$ has no maximum or minimum.**

TRUE *If $f'(x)$ is positive on an interval, then $f(x)$ is increasing on that interval.* **Yes, this follows from the mean value theorem.**

FALSE *The function $\frac{1}{3}x^3$ is the only antiderivative of x^2.* **No, anything of the form $\frac{1}{3}x^3 + C$ would work.**

Problem 9 (10 pts) *For the function below, find the critical points and identify which are relative minima and which are relative maxima.*

$$f(x) = x^3 + 50x^2 + 20$$

We take the derivative and set it equal to zero, and solve:

$$\begin{aligned} f'(x) &= 3x^2 + 100x = 0 \\ x(3x+100) &= 0 \end{aligned}$$

So either $x = 0$ or $x = -\frac{100}{3}$. These are the two critical points.

Now we take the second derivative:

$$f''(x) = 6x + 100$$

Plugging in the critical points, we have

$$f''(0) = 6 \cdot 0 + 100 = 100 > 0$$

so 0 is a local minimum. We plug in $-100/3$:

$$f''(-100/3) = 6 \cdot -\frac{100}{3} + 100 = -100 < 0$$

so $-100/3$ is a local maximum.

Problem 10 (10 pts) *Find the absolute maximum of the function*

$$f(x) = x^2 + 4x + 3$$

over the interval $-3 \leq x \leq 1$.

We first find critical points:

$$f'(x) = 2x + 4 = 0$$

Solving this leads to $x = -2$. So take the critical point $x = -2$ and the endpoints $x = -3$ and $x = 1$ and these are the possible locations for the absolute maximum. We know the absolute maximum exists because the domain is closed and bounded by the extreme value theorem.

We plug into $f(x)$ to see which is the absolute maximum:

$$\begin{aligned} f(-2) &= (-2)^2 + 4(-2) + 3 = 4 - 8 + 3 = -1 \\ f(-3) &= (-3)^2 + 4(-3) + 3 = 9 - 12 + 3 = 0 \\ f(1) &= 1^2 + 4(1) + 3 = 1 + 4 + 3 = 8 \end{aligned}$$

Since the largest value, 8, occurs at $x = 1$, the absolute maximum is at $x = 1$.

Problem 11 (10 pts) *Determine for which values of x the following function is concave up.*

$$f(x) = x^4 - x^2 + 3x + 5$$

This is when $f''(x) > 0$. So we find the second derivative:

$$\begin{aligned} f'(x) &= 4x^3 - 2x + 3 \\ f''(x) &= 12x^2 - 2 \end{aligned}$$

We then need to solve the inequality

$$12x^2 - 2 > 0$$

We can do this in the following way:

$$\begin{aligned} 12x^2 &> 2 \\ x^2 &> \frac{1}{6} \end{aligned}$$

This occurs when $x > \sqrt{1/6}$ or when $x < -\sqrt{1/6}$. That is when f is concave up.

Problem 12 (15 pts) *A container in the shape of a right circular cylinder with no top has surface area of 75π ft^2. What height and radius will maximize volume? You must also justify why this is the absolute maximum. Note: Area of the curved side of a cylinder is base circumference times height, and volume of the cylinder is area of the base times height.*

We are maximizing the volume V of the cylinder. We control r, the radius of the base in feet, and h, the height of the cylinder in feet. There is a constraint

$$75\pi = 2\pi rh + \pi r^2$$

that comes from considering the area of the side plus the area of the base.

We can solve this for h:

$$\begin{aligned} 75\pi &= 2\pi rh + \pi r^2 \\ 75\pi - \pi r^2 &= 2\pi rh \\ \frac{75\pi - \pi r^2}{2\pi r} &= h \end{aligned}$$

Now we are maximizing volume, V, which is

$$V = \pi r^2 h.$$

We substitute using the formula we obtained for h:

$$V = \pi r^2 \frac{75\pi - \pi r^2}{2\pi r}$$

and simplify:

$$\begin{aligned} V &= \pi r^2 \frac{75\pi - \pi r^2}{2\pi r} \\ &= r\frac{75\pi - \pi r^2}{2} \\ &= \frac{75\pi}{2} r - \frac{\pi}{2} r^3 \end{aligned}$$

To find where V is maximized, we differentiate and set equal to 0:

$$\begin{aligned}
\frac{dV}{dr} &= \frac{75\pi}{2} - \frac{3\pi}{2}r^2 = 0 \\
\frac{75\pi}{2} &= \frac{3\pi}{2}r^2 \\
75\pi &= 3\pi r^2 \\
25 &= r^2 \\
r &= \pm 5
\end{aligned}$$

Now $r = -5$ is not in the domain (in this problem, $r \geq 0$), so the only critical point is $r = 5$.

We can check to see if it is a relative maximum using the second derivative test:

$$V''(t) = -3\pi r$$

$$V''(5) = -15\pi < 0$$

Since this is negative, we know that this is a relative maximum.

But we need to see if this is a absolute maximum. To check this, there are several methods. For instance, we look at what happens as r approaches 0 and $+\infty$. For $r = 0$ we see $V(0) = 0$, and $\lim_{r\to\infty} V(r) = -\infty$. Compared with $V(5) = 125\pi$, we see that the absolute maximum occurs at $r = 5$.

We plug back into the formula for h:

$$h = \frac{75\pi - \pi 5^2}{2\pi 5} = \frac{50\pi}{10\pi} = 5$$

So the cylinder has a radius of 5 feet and a height of 5 feet.

Problem 13 (5 pts) *A tank is being filled with water at a rate of*

$$8t + 5$$

liters per second, where t is the time in seconds. Find a formula for how much water is in the tank as a function of t, if we know that at $t = 1$, there are 12 liters in the tank.

If $w(t)$ is the amount of water in the tank, then we know

$$\frac{dw}{dt} = 8t + 5$$

We identify the general antiderivative to determine $w(t)$:

$$w(t) = 4t^2 + 5t + C$$

We know $w(1) = 12$, so we plug in and see

$$12 = 4(1)^2 + 5 \cdot 1 + C$$

and simplifying, we get

$$12 = 9 + C$$

and we get $C = 3$. Therefore

$$w(t) = 4t^2 + 5t + 3$$

Problem 14 (5 pts) *Give an example of a function that has one root, and a starting x value for which Newton's method results in the first iteration getting further from the root than the original x value was. You may describe the function using a graph instead of a formula, if you like.*