Math 210 Midterm 5 Answers

Problem 1 (5 pts)

$$\int (x^5 + x^2 + \sin x)\, dx$$

$$\frac{1}{6}x^6 + \frac{1}{3}x^3 - \cos x + C$$

Problem 2 (5 pts)

$$\int_0^2 3^x\, dx$$

$$\begin{aligned} &= \frac{1}{\ln 3}3^x\Big|_0^2 \\ &= \frac{1}{\ln 3}3^2 - \frac{1}{\ln 3}3^0 \\ &= \frac{9}{\ln 3} - \frac{1}{\ln 3} \\ &= \frac{8}{\ln 3} \end{aligned}$$

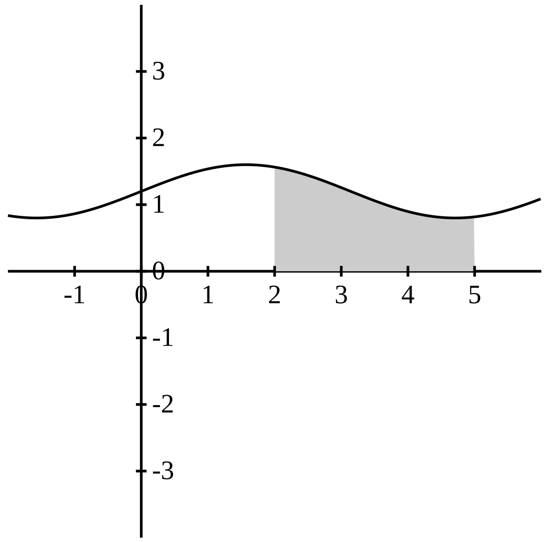
Problem 3 (5 pts)

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\begin{aligned} &= \sin^{-1} x\big|_0^{1/2} \\ &= \sin^{-1}(1/2) - \sin^{-1}(0) \\ &= \frac{\pi}{6} - 0 = \frac{\pi}{6} \end{aligned}$$

Problem 4 (5 pts) *For the function $f(x)$ graphed below, shade the region whose area corresponds to*

$$\int_2^5 f(x)\, dx$$



Problem 5 (5 pts) *Find*

$$\int_1^2 \sin(x^{10}) 10x^9 \, dx$$

We recognize that the integrand looks like an application of a chain rule, namely to $-\cos(x^{10})$.

$$\begin{aligned} &= -\cos(x^{10})\big|_1^2 \\ &= -\cos(2^{10}) - (-\cos(1^{10})) \\ &= -\cos(1024) + \cos(1) \end{aligned}$$

Problem 6 (5 pts) *Find*

$$\int_2^7 f(x) \, dx$$

where

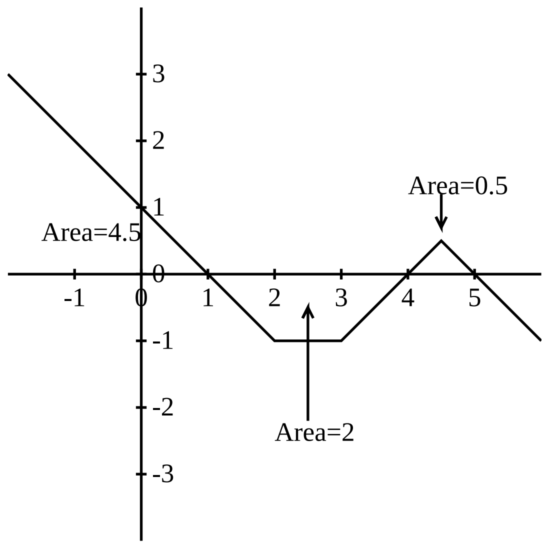
$$f(x) = \begin{cases} 9, & x > 3 \\ x^2, & x \leq 3 \end{cases}$$

$$\begin{aligned} &= \int_2^3 f(x)\,dx + \int_3^7 f(x)\,dx \\ &= \int_2^3 x^2\,dx + \int_3^7 9\,dx \\ &= \frac{1}{3}x^3\bigg|_2^3 + 9x|_3^7 \\ &= \left(\frac{1}{3}3^3 - \frac{1}{3}2^3\right) + (9 \cdot 7 - 9 \cdot 3) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{27}{3} - \frac{8}{3}\right) + (63 - 27) \\
&= \frac{19}{3} + 36 \\
&= \frac{127}{3}
\end{aligned}
$$

Problem 7 (10 pts) *Below is a picture, with various areas labeled. Find*

$$\int_1^5 f(x)\, dx$$



The integral uses the area 2 but with a minus sign, and the area 0.5 with a positive sign.

$$-2 + 0.5 = -1.5$$

Problem 8 (10 pts) *Find the area between the x axis and $f(x) = \sec x \cdot \tan x$ between $x = 0$ and $x = 1$.*

$$
\begin{aligned}
&= \int_0^1 \sec x \cdot \tan x \, dx \\
&= \sec x\big|_0^1 \\
&= \sec 1 - \sec 0 \\
&= \sec 1 - 1
\end{aligned}
$$

Problem 9 (10 pts) *Make an estimate of the area under $y = x^2 + 2$ between $x = 1$ and $x = 2$ using $n = 2$ rectangles, computing using the left endpoints.*

We have $a = 1$ and $b = 2$ and $n = 2$. That means $\Delta x = \frac{b-a}{n} = 0.5$, and $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$. We therefore have two intervals: $[1, 1.5]$ and $[1.5, 2]$. Taking the left endpoints we have $x_1^* = 1$, $x_2^* = 1.5$. We find the rectangles:

interval	left endpoint x_i^*	height $f(x_i^*)$	width Δx	area $f(x_i^*)\Delta x$
$[1, 1.5]$	1	$1^2 + 2 = 3$	0.5	1.5
$[1.5, 2]$	1.5	$1.5^2 + 2 = 4.25$	0.5	2.125

We add up the areas and get

$$1.5 + 2.125 = 3.625$$

Problem 10 (10 pts) *The following are Riemann sums for a particular continuous function $f(x)$ that is always increasing. The sums start from $a = 2$ to $b = 9$. The choices for n and x_i^* are on the left. Match them to plausible answers on the right.*

$n = 5$, *left sum*	*3.295*
$n = 5$, *right sum*	*2.353*
$n = 100$, *left sum*	*5.901*
$n = 100$, *right sum*	*3.472*
$n = 100$, *midpoint*	*3.378*

For an increasing function, the left sums are always an underestimate and the right sums are always an overestimate. We expect the answers for $n = 100$ to be much closer to the true answer than the answers for $n = 5$. In other words, the answers for $N = 5$ should be the farthest from the true answer. So $n = 5$ left sum should be the lowest answer, 2.353; and $n = 5$ right sum should be the highest answer, 5.901.

Among the $n = 100$ estimates, the left sum should be the lowest, the right sum should be the highest, and the midpoint should be in between. So $n = 100$ left sum should be 3.295, $n = 100$ midpoint should be 3.378, and $n = 100$ right sum should be 3.472.

Problem 11 (10 pts) *True or false:*

TRUE *Every continuous function has at least one antiderivative.* **By Part 1 of the Fundamental Theorem of Calculus, an antiderivative of $f(x)$ is $\int_0^x f(t)\,dt$.**

FALSE *A definite integral is always positive.* **It can be negative if the function goes below the x axis or if the integral goes from right to left.**

FALSE *The actual area under a curve is always exactly half-way between the left Riemann sum and the right Riemann sum.* **Although under many circumstances we expect it to be somewhere between the two, we do not know in general if it is exactly half way between the two.**

TRUE *As the number n of rectangles increases to infinity, the left Riemann sum and the right Riemann sum both converge to the same answer.* **Yes, this is the idea of Riemann sums as a way to approach the definite integral.**

FALSE *The derivative of* $\int_3^5 f(x)\,dx$ *is always* $f(x)$. **No,** $\int_3^5 f(x)\,dx$ **is just a number. If you view it as a constant, the derivative is 0.**

Problem 12 (5 pts) *Find*

$$\frac{d}{dx}\int_3^x \frac{3t^2+9}{5t+\sin t}\,dt$$

Using the Fundamental Theorem of Calculus (part 1), we get

$$\frac{3x^2+9}{5x+\sin x}$$

Problem 13 (5 pts) *Find the derivative of*

$$\int_{3x}^{x^2} \tan(t)\,dt$$

We first rewrite this as:

$$\begin{aligned} &= \int_{3x}^{1} \tan(t)\,dt + \int_1^{x^2} \tan(t)\,dt \\ &= -\int_1^{3x} \tan(t)\,dt + \int_1^{x^2} \tan(t)\,dt \end{aligned}$$

Then we take the derivative using the Fundamental Theorem of Calculus (part 1), and the chain rule:

$$-\tan(3x)\cdot 3 + \tan(x^2)(2x)$$

Problem 14 (5 pts) *The following is a Riemann sum. Express it as a definite integral from* $a = 6$ *to* $b = 12$. *(No need to compute the value of this definite integral.)*

$$\lim_{n\to\infty}\sum_{i=1}^{n} x_i^3 \ln(1+x_i)\Delta x$$

$$\int_6^{12} x^3 \ln(1+x)\,dx$$

Problem 15 (5 pts) *The functions f and g are given in the tables below. We know that $f'(x) = g(x)$. Find the most accurate answer you can to the following:*

$$\int_2^5 g(x)\,dx$$

x	$f(x)$	$g(x)$
0	3	2
1	5	2.5
2	8	4
3	15	6
4	19	3
5	22	2
6	24	1

Using the Fundamental Theorem of Calculus, we know that

$$\int_2^5 g(x)\,dx = f(5) - f(2)$$

since $f'(x) = g(x)$. We can then plug in values from the table to get

$$f(5) - f(2) = 22 - 8 = 14.$$

This is an exact answer.

Note that it is also possible to estimate this answer by using Riemann sums on $g(x)$. The most number of points awarded in that case will be 3 out of 5 since it is not the most accurate answer possible.